

# AN EXPLICIT FORMULA FOR THE BEREZIN STAR PRODUCT

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**ABSTRACT.** We prove an explicit formula of the Berezin star product on Kähler manifolds. The formula is expressed as a summation over certain strongly connected digraphs. The proof relies on a combinatorial interpretation of Engliš' work on the asymptotic expansion of the Laplace integral.

## 1. INTRODUCTION

Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [1, 2] introduced quantization as a deformation of the usual commutative product into a noncommutative associative  $\star$ -product. The existence of  $\star$ -products on symplectic manifolds was solved by De Wilde and Lecomte [17], Omori, Maeda and Yoshioka [45, 46], and Fedosov [24]. The most spectacular result on the existence of  $\star$ -products was Kontsevich's proof [35] of a universal formula that gives a  $\star$ -product on any Poisson manifold.

Let  $(M, g)$  be a Kähler manifold of dimension  $n$ . On a coordinate chart  $\Omega$ , the Kähler form is given by

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_{\bar{j}}.$$

If  $\Omega$  is contractible, there exists a Kähler potential  $\Phi$  satisfying

$$\partial\bar{\partial}\Phi = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_{\bar{j}}.$$

Recall that around each point  $x \in M$ , there exists a normal coordinate system such that

$$(1) \quad g_{i\bar{j}}(x) = \delta_{ij}, \quad g_{i\bar{j}k_1 \dots k_r}(x) = g_{i\bar{j}l_1 \dots l_r}(x) = 0$$

for all  $r \leq N \in \mathbb{N}$ , where  $N$  can be chosen arbitrary large and  $g_{i\bar{j}k_1 \dots k_r} = \partial_{k_1 \dots k_r} g_{i\bar{j}}$ .

The Poisson bracket of the functions  $f_1, f_2 \in C^\infty(M)$  can be expressed locally as

$$(2) \quad \{f_1, f_2\} = ig^{k\bar{l}} \left( \frac{\partial f_1}{\partial z^k} \frac{\partial f_2}{\partial \bar{z}^l} - \frac{\partial f_2}{\partial z^k} \frac{\partial f_1}{\partial \bar{z}^l} \right).$$

Let  $C^\infty(M)[[h]]$  denote the algebra of formal power series in  $h$  over  $C^\infty(M)$ . A star product is an associative  $C[[h]]$ -bilinear product  $\star$  such that  $\forall f_1, f_2 \in C^\infty(M)$ ,

$$(3) \quad f_1 \star f_2 = \sum_{j=0}^{\infty} h^j C_j(f_1, f_2),$$

where the  $\mathbb{C}$ -bilinear operators  $C_j$  satisfy

$$(4) \quad C_0(f_1, f_2) = f_1 f_2, \quad C_1(f_1, f_2) - C_1(f_2, f_1) = i\{f_1, f_2\}.$$

There are constructions of  $\star$ -products on restricted types of Kähler manifolds by Berezin [3, 4], Moreno and Ortega-Navarro [42, 43], and Cahen, Gutt and Rawnsley [9, 10, 11]. The existence and classification of deformation quantizations with “separation of variables” for all Kähler manifolds was shown by Karabegov [29]. There are alternative constructions by Bordemann and Waldmann [6], Reshetikhin and Takhtajan [47], Schlichenmaier [48], Engliš [21] and Gammelgaard [28]. See also the recent preprints of Karabegov [32, 33].

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*Key words and phrases.* Berezin star product, Berezin transform, Weyl invariants.

**MSC(2010)** 53D55 32J27.

Let  $\Phi(x, y)$  be an almost analytic extension of  $\Phi(x)$  to a neighborhood of the diagonal, i.e.  $\bar{\partial}_x \Phi$  and  $\partial_y \Phi$  vanish to infinite order for  $x = y$  (cf. [8]). We can assume  $\overline{\Phi(x, y)} = \Phi(y, x)$ . Consider the real valued function

$$D(x, y) = \Phi(x, x) + \Phi(y, y) - \Phi(x, y) - \Phi(y, x),$$

which is called the Calabi diastatic function [12]. It is easily seen that in a sufficiently small neighborhood of the diagonal,  $D(x, y) \geq 0$  and  $D(x, y) = 0$  if and only if  $x = y$ .

For  $\alpha > 0$ , consider the weighted Bergman space of all holomorphic function on  $\Omega$  square-integrable with respect to the measure  $e^{-\alpha \Phi} \frac{w^n}{n!}$ .

We denote by  $K_\alpha(x, y)$  the reproducing kernel. Locally, it is often the case that  $K_\alpha(x, y)$  has an asymptotic expansion in a small neighborhood of the diagonal (see [3, 19, 34]) for  $\alpha \rightarrow \infty$ ,

$$(5) \quad K_\alpha(x, y) \sim e^{\alpha \Phi(x, y)} \sum_{k=0}^{\infty} B_k(x, y) \alpha^{-k}.$$

To ensure the convergence of this expansion, we could take the sufficiently small neighborhood  $\Omega$  to be a strongly pseudoconvex domain with real analytic boundary (cf. [19, 20]). We will implicitly take such  $\Omega$  throughout the paper, which is needed to guarantee the convergence of other asymptotic expansions such as (7). The Bergman kernels  $B_k(x, x)$  on the diagonal turn out to be linear combinations with universal coefficients of Weyl invariants (see Section 2 for the definition) independent of  $\Omega$ . For discussions on the convergence in the compact Kähler case, see [34].

The Bergman kernel  $B_k$  in the setting when  $\Omega$  is a compact Kähler manifold was also much studied (cf. [13, 50, 53, 38, 37]). Dai, Liu and Ma [16] proved the most general version for the asymptotic expansion of Bergman kernels on orbifolds and symplectic manifolds.

The *Berezin transform* is the integral operator

$$(6) \quad I_\alpha f(x) = \int_{\Omega} f(y) \frac{|K_\alpha(x, y)|^2}{K_\alpha(x, x)} e^{-\alpha \Phi(y)} \frac{w_g^n(y)}{n!}.$$

At any point for which  $K_\alpha(x, x)$  invertible, the integral converges for each bounded measurable function  $f$  on  $\Omega$ . Note that (5) implies that for any  $x$ ,  $K_\alpha(x, x) \neq 0$  if  $\alpha$  is large enough.

The Berezin transform has an asymptotic expansion for  $\alpha \rightarrow \infty$  (cf. [19, 34]),

$$(7) \quad I_\alpha f(x) = \sum_{k=0}^{\infty} Q_k f(x) \alpha^{-k},$$

where  $Q_k$  are linear differential operators.

The Berezin star product was introduced by Berezin [3] through symbol calculus for linear operators on weighted Bergman spaces (cf. [21]). As noted by Karabegov [30], the Berezin star product is related to the asymptotic expansion of the Berezin transform in a nice way (cf. [22]). Denote by  $c_{j\alpha\beta} \partial^\alpha \bar{\partial}^\beta$  the coefficients of  $Q_j$ ,

$$(8) \quad Q_j f = \sum_{\alpha, \beta \text{ multiindices}} c_{j\alpha\beta} \partial^\alpha \bar{\partial}^\beta f.$$

Then the coefficients of the Berezin star product are given by bilinear differential operators

$$(9) \quad C_j(f_1, f_2) := \sum_{\alpha, \beta} c_{j\alpha\beta} (\bar{\partial}^\beta f_1) (\partial^\alpha f_2).$$

The Berezin star product is equivalent to the Berezin-Toeplitz star product  $\star_{BT}$  via the Berezin transform (cf. [34])

$$(10) \quad f_1 \star_{BT} f_2 = I^{-1} (I f_1 \star I f_2),$$

where  $I := I_{1/h}$  is obtained from substituting  $\alpha$  by  $1/h$  in  $I_\alpha$ .

Recall that the Toeplitz operator  $T_f^{(m)}$  for  $f \in C^\infty(M)$  is defined to be

$$(11) \quad T_f^{(m)} := \Pi^{(m)}(f \cdot) : H^0(M, L^m) \rightarrow H^0(M, L^m),$$

where  $\Pi^{(m)} : L^2(M, L^m) \rightarrow H^0(M, L^m)$  is the projection.

It was proved by Schlichenmaier [48] that the Berezin-Toeplitz star product (10) is the unique star product

$$(12) \quad f_1 \star_{BT} f_2 := \sum_{j=0}^{\infty} h^j C_j^{BT}(f_1, f_2),$$

such that the following asymptotic expansion holds

$$(13) \quad T_{f_1}^{(m)} T_{f_2}^{(m)} \sim \sum_{j=0}^{\infty} m^{-j} T_{C_j^{BT}(f_1, f_2)}^{(m)}, \quad m \rightarrow \infty.$$

The Berezin transform was introduced by Berezin [4] for symmetric domains in  $\mathbb{C}^n$ . Among the pioneers in extending Berezin's results are Unterberger and Upmeyer [51], Engliš [18], and Engliš and Peetre [23]. The coefficients  $Q_i$ ,  $i \leq 3$  have been obtained by Engliš [19]. Karabegov and Schlichenmaier [34] proved the asymptotic expansion of the Berezin transform for compact Kähler manifolds and showed that the Berezin-Toeplitz deformation quantization has the property of separation of variables. Ma-Marinescu [41, 40, Ch.7] developed the theory of Toeplitz operators on symplectic manifolds in the presence of a twisting vector bundle and showed that the calculation of the coefficients of the Berezin-Toeplitz star product is local. Moreover, in the Kähler case, Ma-Marinescu [39] calculated the first three terms of the expansion of the kernels of Toeplitz operators and the Berezin-Toeplitz star product with a twisting vector bundle. (cf. also [5, 7, 14, 21, 22, 27, 31] and especially the nice survey by Schlichenmaier [49] for related works on Berezin-Toeplitz quantization).

Using the closed graph-theoretic formula (41), we computed  $Q_k$ ,  $k \leq 4$  in Example 4.1 and the appendix. We also computed the first four coefficients of the Berezin-Toeplitz star product in Example 4.3.

As discussed in Section 2, we can write  $Q_k f$  as a sum over graphs.

$$(14) \quad Q_k f = \sum_{\Gamma \in \dot{\mathcal{G}}(k)} Q_{\Gamma} \Gamma, \quad k \geq 0.$$

The closed formula (41) of  $Q_{\Gamma}$  leads to the main result of this paper, which is the following explicit formula for the Berezin star product.

**Theorem 1.1.** *Let  $M$  be a Kähler manifold. Fix a normal coordinate system around  $x \in M$ , the following equation for the Berezin  $\star$ -product holds at  $x$ .*

$$(15) \quad f_1 \star f_2(x) = \sum_{\Gamma=(V \cup \{f\}, E) \in \dot{\mathcal{G}}_{scn}} \frac{\det(A(\Gamma_-) - I)}{|\text{Aut}(\Gamma)|} h^{|E|-|V|} D_{\Gamma}(f_1, f_2) \Big|_x,$$

where  $\Gamma$  runs over the set of strongly connected pointed stable graphs;  $\Gamma_-$  is the subgraph of  $\Gamma$  obtained by removing the distinguished vertex  $f$  from  $\Gamma$  and  $A(\Gamma_-)$  is its adjacency matrix.  $D_{\Gamma}(f_1, f_2)$  is the partition function of  $\Gamma$  (see Definition 2.5 in Section 2). Moreover, we can convert the right-hand side to the invariant tensor expression easily.

The expression of the Berezin star product as a summation over graphs is not surprising as there are remarkable pioneering works by Kontsevich [35], and Reshetikhin and Takhtajan [47]. However, the formula (15) may be the simplest concerning the star-products.

Kontsevich's universal formula of a star product on Poisson manifolds was written as a summation over labeled directed graphs with two distinguished vertices and the coefficients are certain integrals over configuration spaces. The Feynmann graph formula of Reshetikhin and Takhtajan is for the non-normalized Berezin star product (1 is not the unit in the product) on arbitrary Kähler manifolds. In an attempt to normalize Reshetikhin and Takhtajan's formula, Gammelgaard [28] obtained a universal formula for any star product with separation of variables corresponding to a given classifying Karabegov form. His formula is expressed as a summation over weighted acyclic graphs. As pointed out by a referee, Gammelgaard's formula does not allow to express the Berezin star product directly since the Karabegov form of the Berezin star product is not explicitly known in general. In the quantizable compact Kähler case, the Karabegov form of the Berezin star product was given in terms

of the Bergman kernel evaluated along the diagonal (cf. [49, §7]) and the Karabegov form of the Berezin-Toeplitz star product was also identified in [34].

Although Berezin transform (6) can be defined only under very restrictive conditions, the coefficients  $Q_k$  of its asymptotic expansion, as long as it exists, are universal polynomials of curvature tensor independent of the underlying Kähler metric. As a result, the formula (15) defines a universal (associative) star product with separation of variables on an arbitray Kähler manifold. A proof will be given at the end of §3.

**Acknowledgements** The author is grateful to Professor Kefeng Liu, whose help and guidance saved my math career. The author thanks the referees for very helpful comments and suggestions.

## 2. WEYL INVARIANTS AND GRAPHS

A *digraph* (directed graph)  $G = (V, E)$  is defined to be a finite set  $V$  (whose elements are called vertices) and a multiset  $E$  of ordered pairs of vertices, called directed edges. Throughout the paper, we allow a digraph to have loops and multi-edges. The indegree and outdegree of a vertex  $v$  are denoted by  $\deg^-(v)$  and  $\deg^+(v)$  respectively. The adjacency matrix  $A = A(G)$  of a digraph  $G$  with  $n$  vertices is a square matrix of order  $n$  whose entry  $A_{ij}$  is the number of directed edges from vertex  $i$  to vertex  $j$ .

A digraph  $G$  is called *connected* if the underlying undirected graph is connected, and *strongly connected* if there is a directed path from each vertex in  $G$  to every other vertex. For a digraph  $G = (V, E)$ , we can partition  $V$  into strongly connected components, namely the maximal strongly connected subgraphs of  $G$ . Among these strongly connected components, we have at least one *sink* (a component without outgoing edges) and one *source* (a component without incoming edges).

A *full subdigraph*  $H$  of a digraph  $G$  is a digraph whose vertex set is a subset of the vertex set of  $G$ , the set of edges connecting any two vertices  $v_1$  and  $v_2$  in  $H$  equals the set of edges connecting  $v_1$  and  $v_2$  in  $G$ .

Given two subgraphs  $G_1$  and  $G_2$  of  $G$ , we denote by  $G_1 + G_2$  the full subgraph of  $G$  whose vertices are the vertices of  $G_1$  and  $G_2$ .

Recall the definition of stable and semistable graphs in the previous paper [52]. These graphs were used to represent Weyl invariants, which include the coefficients of the asymptotic expansion of the Bergman kernel.

**Definition 2.1.** We call a vertex  $v$  of a digraph  $G$  semistable if we have

$$\deg^-(v) \geq 1, \deg^+(v) \geq 1, \deg^-(v) + \deg^+(v) \geq 3.$$

$G$  is called *semistable* if each vertex of  $G$  is semistable. We call  $v$  *stable* if  $\deg^-(v) \geq 2, \deg^+(v) \geq 2$ . A digraph  $G$  is *stable* if each vertex of  $G$  is stable. The set of semistable and stable graphs will be denoted by  $\mathcal{G}^{ss}$  and  $\mathcal{G}$  respectively. For any  $G = (V, E) \in \mathcal{G}^{ss}$ , its weight  $\omega(G)$  is defined to be the integer  $|E| - |V|$ . We denote by  $\mathcal{G}^{ss}(k)$  and  $\mathcal{G}(k)$  respectively the set of semistable and stable digraphs with weight  $k$ . Let  $\mathcal{G}_{con}(k)$  and  $\mathcal{G}_{scon}(k)$  respectively be the set of connected and strongly connected graphs in  $\mathcal{G}(k)$ .

We need to make slight extension of these concepts, in order to write the Berezin star product as a summation over graphs.

**Definition 2.2.** A *(one-)pointed semistable (stable) graph*  $\Gamma = (V \cup \{f\}, E)$  is defined to be a digraph with a distinguished vertex labeled by  $f$ , such that each ordinary vertex  $v \in V$  is semistable (stable). We denote by  $\text{Aut}(\Gamma)$  all automorphisms of the pointed graph  $\Gamma$  fixing the vertex  $f$ . Let  $\Gamma_-$  denote the subgraph of  $\Gamma$  obtained by removing the distinguished vertex  $f$  from  $\Gamma$  and  $A(\Gamma_-)$  its adjacency matrix.

The set of pointed semistable and stable graphs will be denoted by  $\dot{\mathcal{G}}^{ss}$  and  $\dot{\mathcal{G}}$  respectively. For any  $\Gamma \in \dot{\mathcal{G}}^{ss}$ , its weight  $w(\Gamma)$  is defined to be the integer  $|E| - |V|$ . We denote by  $\dot{\mathcal{G}}^{ss}(k)$  and  $\dot{\mathcal{G}}(k)$  respectively the set of pointed semistable and pointed stable digraphs with weight  $k$ . Let  $\dot{\mathcal{G}}_{con}(k)$

$(\dot{\mathcal{G}}_{con}^{ss}(k))$  and  $\dot{\mathcal{G}}_{con}(k)$  ( $\dot{\mathcal{G}}_{con}^{ss}(k)$ ) respectively be the set of connected and strongly connected graphs in  $\dot{\mathcal{G}}(k)$  ( $\dot{\mathcal{G}}^{ss}(k)$ ). We also define a special set of graphs:

$$\dot{\Lambda}(k) = \{\Gamma \in \dot{\mathcal{G}}_{con}(k) \mid 1 \text{ is not an eigenvalue of } A(\Gamma_-)\}.$$

The cardinality of this set is the number of terms in  $Q_k$  by Theorem 3.4. We have computed the cardinalities of these sets when  $k \leq 5$  in Table 1.

TABLE 1. Numbers of pointed stable graphs

$k$	0	1	2	3	4	5
$ \dot{\mathcal{G}}(k) $	1	2	9	46	314	2638
$ \dot{\mathcal{G}}_{con}(k) $	1	1	4	23	178	1637
$ \dot{\mathcal{G}}_{con}^{ss}(k) $	1	1	2	9	61	538
$ \dot{\Lambda}(k) $	1	1	1	5	36	331

**Remark 2.3.** Note that  $\mathcal{G}^{ss}$  ( $\mathcal{G}$ ) may be regarded as a subset of  $\dot{\mathcal{G}}^{ss}$  ( $\dot{\mathcal{G}}$ ) where the distinguished vertex  $f$  is an isolated vertex without loops.

**Remark 2.4.** For later use, we may extend the above definition to semistable (stable) graphs with  $m$  distinguished vertices  $\Gamma = (V \cup \{f_1, \dots, f_m\}, E)$ . We denote by  $\tilde{\mathcal{G}}$  ( $\tilde{\mathcal{G}}^{ss}$ ) the set of semistable (stable) graphs with any number of distinguished vertices. The automorphism of a graph  $\Gamma \in \tilde{\mathcal{G}}$  ( $\Gamma \in \tilde{\mathcal{G}}^{ss}$ ) is always assumed to fix each distinguished vertex and its automorphism group is simply denoted by  $\text{Aut}(\Gamma)$ .

The concept of *Weyl invariants* was introduced by Fefferman [26]. We slightly extend the definition to allow additional functions. Consider the tensor products of covariant derivatives of the curvature tensor  $R_{i\bar{j}k\bar{l};p\cdots\bar{q}}$  and a function  $f$ , e.g.

$$R_{ijkl;p\bar{q}} \otimes \cdots \otimes R_{abcd;\bar{e}} \otimes f_{rst}.$$

The Weyl invariants are constructed by first pairing up the unbarred indices to barred indices and then contracting all paired indices.

As remarked in [52], we can write Weyl invariants as polynomials of  $g_{i\bar{j}\alpha}$  and  $f$  and their derivatives. The advantage is that we do not need to deal with the problem of exchanging indices, thus we can canonically represent a Weyl invariant as a sum over pointed stable graphs, from which we can easily recover its curvature-tensor expression.

We represent a digraph as a weighted digraph. The weight of a directed edge is the number of multi-edges. The number attached to a vertex denotes the number of its self-loops. A vertex without loops will be denoted by a small hollow circle  $\circ$ . The distinguished vertex  $f$  is denoted by a solid circle  $\bullet$ . We will drop it if  $f$  is an isolated vertex without loops.

The Weyl invariant  $g_{i\bar{i}k\bar{l}p}g_{j\bar{j}l\bar{k}q}f_{q\bar{p}}$  is depicted in Figure 1 knowing that  $(i, \bar{i})$ ,  $(j, \bar{j})$  etc. are paired indices to be contracted.

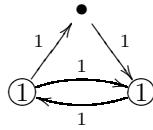


FIGURE 1. The associated graph of  $g_{i\bar{i}k\bar{l}p}g_{j\bar{j}l\bar{k}q}f_{q\bar{p}}$

Below, we may use the same notation to denote a graph and its associated Weyl invariant.

Therefore we can write  $R_k f(x)$  in (28) and  $Q_k f(x)$  in (7) as a weighted sum of pointed semistable graphs in  $\dot{\mathcal{G}}^{ss}(k)$ .

$$(16) \quad R_k f = \sum_{\Gamma \in \dot{\mathcal{G}}^{ss}(k)} R_\Gamma \Gamma, \quad Q_k f = \sum_{\Gamma \in \dot{\mathcal{G}}^{ss}(k)} Q_\Gamma \Gamma.$$

Note that at the center of the normal coordinate, terms of non-stable graphs vanish. Moreover, to recover  $R_\Gamma, Q_\Gamma$  for all semistable pointed graphs  $\Gamma \in \tilde{\mathcal{G}}^{ss}$ , it is enough to know only their values for stable pointed graphs  $\Gamma \in \tilde{\mathcal{G}}$ . We will prove closed formulas of  $R_\Gamma, Q_\Gamma$  in Section 3

**Definition 2.5.** Given  $\Gamma \in \tilde{\mathcal{G}}^{ss}$ , the *partition function*  $D_\Gamma(f_1, f_2)$  is defined to be a Weyl invariant generated from  $\Gamma$  by replacing the vertex  $f$  in with two vertices  $f_1$  and  $f_2$ , such that all inward edges of  $f$  are connected to  $f_1$  and all outward edges of  $f$  are connected to  $f_2$ .

For example, if  $\Gamma$  is the graph in Figure 1, then

$$(17) \quad D_\Gamma(f_1, f_2) = g_{i\bar{i}k\bar{l}p} g_{j\bar{j}l\bar{k}q} \partial_{\bar{p}} f_1 \partial_q f_2.$$

Let  $\Gamma \in \tilde{\mathcal{G}}^{ss}$  and  $H \in \mathcal{G}^{ss}$ , we define  $D_\Gamma(H)$  to be the Weyl invariant generated by replacing  $f$  in  $\Gamma$  with  $H$ . For example, if  $\Gamma$  is the graph in Figure 1, then

$$(18) \quad D_\Gamma(H) = g_{i\bar{i}k\bar{l}p} g_{j\bar{j}l\bar{k}q} \partial_{q\bar{p}} H.$$

The derivatives of  $H$  in the right-hand side may be expanded to get a sum of semistable graphs.

**Remark 2.6.** Given  $\Gamma \in \tilde{\mathcal{G}}$  and two Weyl invariants with associated graphs  $H_1, H_2 \in \mathcal{G}^{ss}$ , we define  $D_\Gamma(H_1, H_2)$  to be sum of stable graphs generated from  $D_\Gamma(f_1, f_2)$  by replacing  $f_1, f_2$  with  $H_1, H_2$ . We emphasize that in this definition, we discard all semistable but not stable graphs. In particular, we only need to consider  $\Gamma$  as acting on vertices of  $H_1, H_2$  but not on edges, since in the latter case, there will appear semistable but not stable vertices (cf. [52, §2]). Note that  $D_\Gamma(H_1, H_2)$  may be defined for  $H_1, H_2 \in \tilde{\mathcal{G}}^{ss}$  similarly.

Let  $\mathcal{L}$  be the set of digraphs consisting of a finite number of vertex-disjoint simple cycles (i.e. simple polygons without common vertex). The length of a simple cycle is defined to be the number of its edges. For each graph  $L \in \mathcal{L}$ , we can write  $L$  as a finite increasing sequence of nonnegative integers  $[i_1, \dots, i_m]$ , meaning  $L$  consists of  $m$  disjoint simple cycles, whose lengths are specified by  $i_1, \dots, i_m$ . We define the index of  $L$  to be

$$(19) \quad i(L) = m + i_1 + \dots + i_m.$$

Note that  $[0]$  is just a single vertex and  $[1]$  is a vertex with a self-loop. If  $0 \notin L$ , then  $L$  is usually called a linear digraph. Recall that a *linear digraph* is a digraph in which  $\deg^+(v) = \deg^-(v) = 1$  for each vertex  $v$ .

Given a set of indices  $\alpha_1, \dots, \alpha_r$ , denote by  $\mathcal{L}(\alpha_1, \dots, \alpha_r)$  the isomorphism classes of all possible decorations of the vertices of  $L \in \mathcal{L}$  with the half-edges  $\alpha_1, \dots, \alpha_r$  requiring each vertex to be semistable. Two decorations of  $L$  that differ by a graph isomorphism are considered the same.

In [52], we proved the following lemma, which explains the graphical properties of the partial derivatives of  $\det g$ .

**Lemma 2.7.** *We have*

$$(20) \quad \frac{1}{\det g} \partial_{\alpha_1} \dots \partial_{\alpha_r} \det g = \sum_{\substack{L \in \mathcal{L}(\alpha_1, \dots, \alpha_r) \\ 0 \notin L}} (-1)^{i(L)} \cdot L.$$

We need the following coefficient theorem (see Theorem 1.2 in [15]) from the spectral graph theory.

**Theorem 2.8.** *Let  $P_G(\lambda) = \lambda^n + c_1 \lambda^{n-1} + \dots + c_n$  be the characteristic polynomial of a digraph  $G$  with  $n$  vertices. Then for each  $i = 1, \dots, n$ ,*

$$c_i = \sum_{L \in \mathcal{L}_i} (-1)^{p(L)},$$

where  $\mathcal{L}_i$  is the set of all linear directed subgraphs  $L$  of  $G$  with exactly  $i$  vertices;  $p(L)$  denotes the number of components of  $L$ .

The following lemma and remark will be used in §3.

**Lemma 2.9.** *Let  $\Gamma \in \dot{\mathcal{G}}$  and  $H_1, H_2 \in \mathcal{G}^{ss}$ . (cf. Remark 2.6 for the definition of  $D_\Gamma(H_1, H_2)$ ) Then*

$$(21) \quad \frac{1}{|\text{Aut}(\Gamma)||\text{Aut}(H_1)||\text{Aut}(H_2)|} D_\Gamma(H_1, H_2) = \sum_G \frac{1}{|\text{Aut}(G)|} G,$$

where  $G$  in the summation runs over isomorphism classes of graphs appearing in the expansion of  $D_\Gamma(H_1, H_2)$ .

*Proof.* Note that the group  $\text{Aut}(\Gamma) \times \text{Aut}(H_1) \times \text{Aut}(H_2)$  has a natural action on the set of all graphs in the expansion of  $D_\Gamma(H_1, H_2)$ . Then it is not difficult to see that the set of orbits corresponds to isomorphism classes of graphs and the isotropy group at a graph  $G$  is just  $\text{Aut}(G)$ . So we get the desired equation.  $\square$

**Remark 2.10.** Let  $\Gamma \in \dot{\mathcal{G}}$  be a (one-)pointed stable graph and  $H_1, H_2 \in \tilde{\mathcal{G}}^{ss}$  semistable graphs with (any number of) distinguished vertices. (cf. Remark 2.4). Then the above lemma still holds without any change.

$$(22) \quad \frac{1}{|\text{Aut}(\Gamma)||\text{Aut}(H_1)||\text{Aut}(H_2)|} D_\Gamma(H_1, H_2) = \sum_G \frac{1}{|\text{Aut}(G)|} G,$$

where  $G$  in the summation runs over isomorphism classes of graphs appearing in the expansion of  $D_\Gamma(H_1, H_2)$ . Note that if  $H_1, H_2$  has  $m_1, m_2$  distinguished vertices respectively, then  $G$  has  $m_1 + m_2$  distinguished vertices.

Finally, we record two identities that can convert covariant derivatives of curvature tensors to partial derivatives of metrics and vice versa.

$$(23) \quad R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}k\bar{l}} + g^{m\bar{p}} g_{m\bar{j}l} g_{ik\bar{p}},$$

$$(24) \quad T_{\beta_1 \dots \beta_q; \gamma}^{\alpha_1 \dots \alpha_p} = \partial_\gamma T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} - \sum_{i=1}^q \Gamma_{\gamma \beta_i}^\delta T_{\beta_1 \dots \beta_{i-1} \delta \beta_{i+1} \dots \beta_q}^{\alpha_1 \dots \alpha_p} + \sum_{j=1}^q \Gamma_{\delta \gamma}^{\alpha_j} T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_{j-1} \delta \alpha_{j+1} \dots \alpha_p}.$$

The second equation gives the formula for covariant derivatives of a tensor field  $T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$ , where the Christoffel symbols  $\Gamma_{\beta \gamma}^\alpha = 0$  except for  $\Gamma_{jk}^i = g^{i\bar{l}} g_{j\bar{l}k}$ ,  $\Gamma_{\bar{j}\bar{k}}^{\bar{i}} = g^{\bar{l}i} g_{l\bar{j}\bar{k}}$ .

### 3. THE ASYMPTOTIC EXPANSION OF THE BEREZIN TRANSFORM

We may write (6) as

$$(25) \quad \sum_{k=0}^{\infty} B_k(x) \alpha^{n-k} I_\alpha f(x) = \int_{\Omega} f(y) |K_\alpha(x, y)|^2 e^{-\alpha \Phi(x) - \alpha \Phi(y)} \frac{w_g^n(y)}{n!}.$$

By (5), we have that for  $(x, y)$  near the diagonal,

$$(26) \quad |K_\alpha(x, y)|^2 e^{-\alpha \Phi(x) - \alpha \Phi(y)} = e^{-\alpha D(x, y)} \sum_{k=0}^{\infty} \sum_{i=0}^k B_i(x, y) B_{k-i}(y, x) \alpha^{n-k}.$$

We need the following important result.

**Theorem 3.1.** (Engliš) *If the Laplace integral*

$$\int_{\Omega} f(y) e^{-m D(x, y)} \frac{\omega_g^n(y)}{n!}$$

*exists for some  $m = m_0$ , then it also exists for all  $m > m_0$ . Moreover, it has an asymptotic expansion for  $m \rightarrow \infty$ ,*

$$(27) \quad \int_{\Omega} f(y) e^{-m D(x, y)} \frac{\omega_g^n(y)}{n!} \sim \frac{1}{m^n} \sum_{j \geq 0} m^{-j} R_j(f)(x),$$

where  $R_j : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$  are explicit differential operators defined by

$$(28) \quad R_j f(x) = \frac{1}{\det g} \sum_{k=j}^{3j} \frac{1}{k!(k-j)!} L^k(f \det g S^{k-j})|_{y=x},$$

where  $L$  is the (constant-coefficient) differential operator

$$L f(y) = g^{i\bar{j}}(x) \partial_i \partial_{\bar{j}} f(y)$$

and the function  $S(x, y)$  satisfies

$$\begin{aligned} S = \partial_\alpha S = \partial_{\alpha\beta} S = \partial_{i_1 i_2 \dots i_m} S = \partial_{\bar{i}_1 \bar{i}_2 \dots \bar{i}_m} S = 0 \quad \text{at } y = x, \\ \partial_{i\bar{j}} \alpha_1 \alpha_2 \dots \alpha_m S|_{y=x} = -\partial_{\alpha_1 \alpha_2 \dots \alpha_m} g_{i\bar{j}}(x). \end{aligned}$$

Formulas for  $R_0$  and  $R_1$  were computed by Berezin [3]. Engliš [19] obtained  $R_k$ ,  $k \leq 3$  by a tour de force computation.

**Theorem 3.2.** *Let  $\Gamma = (V \cup \{f\}, E) \in \dot{\mathcal{G}}^{ss}$ . Then we have*

$$(29) \quad R_\Gamma = \frac{\det(A(\Gamma_-) - I)}{|\text{Aut}(\Gamma)|},$$

where  $\Gamma_-$  is the subgraph of  $\Gamma$  obtained by removing the vertex  $f$  from  $\Gamma$ .

*Proof.* Let  $\mathcal{L}$  be the set of linear directed subgraphs of  $\Gamma_-$ . We define an equivalence relation  $\sim$  on  $\mathcal{L}$  by

$$(30) \quad L_1 \sim L_2 \text{ if there is an automorphism } h \in \text{Aut}(\Gamma) \text{ such that } h(L_1) = L_2.$$

Let  $\tilde{\mathcal{L}} = \mathcal{L} / \sim$  be the equivalence class. Given  $L \in \mathcal{L}$ , let  $\text{Aut}(L)$  be the subgroup of  $\text{Aut}(\Gamma)$  that leaves  $L$  invariant. From Lemma 2.7 and (28), we have

$$(31) \quad R_\Gamma = \sum_{L \in \tilde{\mathcal{L}}} \frac{(-1)^{p(L)+|V|}}{|\text{Aut}(L)|},$$

where  $p(L)$  denotes the number of components of  $L$ .

We have the natural action of  $\text{Aut}(\Gamma)$  on  $\mathcal{L}$ . Then the set of orbits is  $\tilde{\mathcal{L}}$  and the isotropy group at  $L$  is  $\text{Aut}(L)$ . So we get the desired equation (29) from Theorem 2.8.  $\square$

From (25), (26) and Theorem 3.1, we get Loi's recursion formula [37]

$$(32) \quad B_k(x) = - \sum_{\substack{i+j=k \\ i, j \geq 1}} B_i(x) B_j(x) - \sum_{\substack{\ell+i+j=k \\ 1 \leq \ell \leq k}} R_\ell(B_i(x, y) B_j(y, x))|_{y=x}.$$

It was pointed out to the author recently that essentially the same identity was also obtained independently in [14].

In [52],  $B_k$  was written as a summation over graphs.

$$(33) \quad B_k(x) = \sum_{G \in \mathcal{G}^{ss}(k)} z(G) \cdot G, \quad z(G) \in \mathbb{Q}$$

and it is proved that if  $G = (V, E) \in \mathcal{G}^{ss}$  is strongly connected, then

$$(34) \quad z(G) = -\frac{\det(A - I)}{|\text{Aut}(G)|},$$

where  $A$  is the adjacency matrix of  $G$ . This formula also follows directly from Theorem 3.2 and (32).

In general, if  $G \in \mathcal{G}^{ss}$  is a disjoint union of connected subgraphs  $G = G_1 \cup \dots \cup G_n$ , then we have

$$(35) \quad z(G) = \begin{cases} \frac{(-1)^n \det(A - I)}{|\text{Aut}(G)|}, & \text{if all } G_i \text{ are strongly connected;} \\ 0, & \text{otherwise.} \end{cases}$$

The following proposition was proved in [52]. Here we give a shorter proof using Theorem 3.2.



**Proposition 3.3.** *If  $G \in \mathcal{G}^{ss}$  is connected but not strongly connected, then  $z(G) = 0$ .*

*Proof.* We will proceed by induction on the weight of  $G$ . First we assume that  $G \in \mathcal{G}(k)$  is stable and has weight  $k$ . Then any sink or source of  $G$  must be at least semistable. Without loss of generality, we may assume that  $C \in \mathcal{G}^{ss}$  is a sink of  $G$ .

In the right-hand side of (32), graphs from the first summation are disconnected, so they do not contribute to  $z(G)$  and may be omitted. For the second summation in the right-hand side of (32), we denote by  $\tilde{B}^+$  and  $\tilde{B}^-$  the source  $B_j(y, x)$  and the sink  $B_i(x, y)$  respectively. By induction, both  $\tilde{B}^+$  and  $\tilde{B}^-$  are disjoint union of strongly connected semistable graphs. By (29), we see that  $z(G)$  equals the coefficient of  $G$  in

$$(36) \quad - \sum_{\substack{\ell+i+j=k \\ 1 \leq \ell \leq k}} \sum_{\substack{\Gamma \in \mathcal{G}(\ell) \\ \tilde{B}^+ \in \mathcal{G}^{ss}(j), \tilde{B}^- \in \mathcal{G}^{ss}(i)}} \frac{\det(A(\Gamma_-) - I)}{|\text{Aut}(\Gamma)|} z(\tilde{B}^+) z(\tilde{B}^-) D_\Gamma(\tilde{B}^+, \tilde{B}^-).$$

Note that in the above summation, we have either  $C \subset \Gamma_-$  or  $C \subset \tilde{B}^-$ . As illustrated in Figure 2, where each arrow in the graph may represent multiple edges, we have  $(\Gamma_-, \tilde{B}^+, \tilde{B}^-) = (H + C, B^+, B^-)$  in the former case and  $(\Gamma_-, \tilde{B}^+, \tilde{B}^-) = (H, B^+, B^- \amalg C)$  in the latter case.

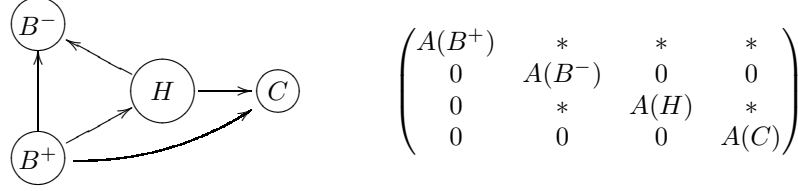


FIGURE 2. A configuration for  $G$  and the adjacency matrix

By Lemma 2.9, the contribution of  $(\Gamma_-, \tilde{B}^+, \tilde{B}^-) = (H + C, B^+, B^-)$  to  $z(G)$  equals

$$(37) \quad -z(B^-)|\text{Aut}(B^-)| \times z(B^+)|\text{Aut}(B^+)| \frac{1}{|\text{Aut}(G)|} \det \left( \begin{pmatrix} A(H) & * \\ 0 & A(C) \end{pmatrix} - I \right) \\ = -z(B^-)|\text{Aut}(B^-)| \times z(B^+)|\text{Aut}(B^+)| \frac{1}{|\text{Aut}(G)|} \det(A(C) - I) \det(A(H) - I)$$

and the contribution of  $(\Gamma_-, \tilde{B}^+, \tilde{B}^-) = (H, B^+, B^- \amalg C)$  to  $z(G)$  equals

$$(38) \quad -z(B^- \amalg C)|\text{Aut}(B^- \amalg C)| \times z(B^+)|\text{Aut}(B^+)| \frac{1}{|\text{Aut}(G)|} \det(A(H) - I) \\ = -z(B^-)|\text{Aut}(B^-)| \times z(C)|\text{Aut}(C)| \times z(B^+)|\text{Aut}(B^+)| \frac{1}{|\text{Aut}(G)|} \det(A(H) - I) \\ = -z(B^-)|\text{Aut}(B^-)| \times (-\det(A(C) - I)) \times z(B^+)|\text{Aut}(B^+)| \frac{1}{|\text{Aut}(G)|} \det(A(H) - I).$$

In the second equation of (38), we used

$$z(B^- \amalg C) = \frac{z(B^-)z(C)}{\varepsilon(B^-, C)}, \\ |\text{Aut}(B^- \amalg C)| = |\text{Aut}(B^-)||\text{Aut}(C)|\varepsilon(B^-, C),$$

where  $\varepsilon(B^-, C) = 1$  or  $2$  depending on whether  $B^- \not\cong C$  or  $B^- \cong C$ .

Therefore for any given  $B^+$  and  $B^-$ , we have that (37) and (38) add up to zero. This concludes the proof of  $z(G) = 0$  when  $G$  is a connected but not strongly connected stable graph.

When  $G$  is only semistable,  $z(G) = 0$  follows from the fact that when using (23) and (24) to convert between covariant derivatives of curvatures and partial derivatives of metrics, we always turn strongly connected graphs into strongly connected graphs.  $\square$

Fix a bounded neighborhood  $U$  of  $x$  such that (26) holds. Applying Theorem 3.1 to the right-hand side of (25) and using (7), we get

$$(39) \quad \sum_{m=0}^k B_m(x) Q_{k-m} f(x) = \sum_{j=0}^k \sum_{i=0}^{k-j} R_j(B_i(x, y) B_{k-j-i}(y, x) f(y))|_{y=x},$$

namely

$$(40) \quad Q_k f(x) = \sum_{j=0}^k \sum_{i=0}^{k-j} R_j(B_i(x, y) B_{k-j-i}(y, x) f(y))|_{y=x} - \sum_{m=1}^k B_m(x) Q_{k-m} f(x),$$

where the operators  $R_j$  apply to the  $y$ -variable.

**Theorem 3.4.** *Let  $\Gamma \in \dot{\mathcal{G}}$ . Then*

$$(41) \quad Q_\Gamma = \begin{cases} \frac{\det(A(\Gamma_-) - I)}{|\text{Aut}(\Gamma)|} & \text{if } \Gamma \text{ is strongly connected,} \\ 0 & \text{otherwise,} \end{cases}$$

where  $\Gamma_-$  is the subgraph of  $\Gamma$  obtained by removing the vertex  $f$  from  $\Gamma$ .

*Proof.* We will use induction on the weight of  $\Gamma$ . There are three cases:

- i) Assume that  $\Gamma \in \dot{\mathcal{G}}$  is a disjoint union of connected subgraphs  $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_n$ ,  $n \geq 1$  and  $\Gamma_1$  is not strongly connected. Since  $\Gamma$  is stable,  $\Gamma_1$  must have a source or sink that does not contain the distinguished vertex  $f$  and belongs to  $\mathcal{G}^{ss}$ .

Using the same argument in the proof of Proposition 3.3, we can prove that the contribution of the first term in the right-hand side of (40) to  $Q_\Gamma$  is zero. The contribution of the second term in the right-hand side of (40) to  $Q_\Gamma$  is also zero by induction and Proposition 3.3.

- ii) Assume that each component in  $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_n$ ,  $n \geq 2$  is strongly connected and  $f \in \Gamma_n$ . Then by Theorem 3.2 and (35), we see that the contribution of the first term in the right-hand side of (40) to  $Q_\Gamma$  is

$$(42) \quad \frac{(-1 - 1 + 1)^{n-1}}{|\text{Aut}(\Gamma)|} \det(A(\Gamma_-) - I).$$

By induction, we see that the contribution of the second term in the right-hand side of (40) to  $Q_\Gamma$  is

$$(43) \quad \frac{(-1)^{n-1}}{|\text{Aut}(\Gamma)|} \det(A(\Gamma_-) - I),$$

which cancel with (42). So combining (i), we have proved that  $Q_\Gamma = 0$  if  $\Gamma$  is not connected.

- iii) If  $\Gamma$  is strongly connected, then the contribution of the second term in the right-hand side of (40) to  $Q_\Gamma$  is 0. the contribution of the first term in the right-hand side of (40) to  $Q_\Gamma$  is just

$$R_\Gamma = \frac{\det(A(\Gamma_-) - I)}{|\text{Aut}(\Gamma)|}.$$

So we conclude the proof with the above three cases.  $\square$

**Corollary 3.5.** *The equation (41) holds also for  $\Gamma \in \dot{\mathcal{G}}^{ss}$ .*

*Proof.* When  $\Gamma \in \dot{\mathcal{G}}^{ss}$  is strongly connected,  $Q_\Gamma = R_\Gamma$  still holds. When  $\Gamma \in \dot{\mathcal{G}}^{ss}$  is connected but not strongly connected,  $Q_\Gamma = 0$  follows by the same reason as stated in the end of proof of Proposition 3.3.  $\square$

**Corollary 3.6.** *Given  $k \geq 0$ . Let  $\Gamma = \left[ \bullet \bigcirc_k \right]$ . Then*

$$(44) \quad R_\Gamma = Q_\Gamma = \frac{1}{k!}.$$

Engliš [19] defined a scalar invariant  $r_k = R_k(1)$ . We see that only graphs with  $\deg^+ f = \deg^- f = 0$  contribute to  $r_k$ .

**Corollary 3.7.** *We have  $r_0 = 1$  and when  $k \geq 1$  (see Remark 2.3)*

$$(45) \quad r_k = \sum_{G \in \mathcal{G}(k)} r(G)G = \sum_{G \in \mathcal{G}(k)} \frac{\det(A(G) - I)}{|\text{Aut}(G)|} G.$$

Thus we have  $r(G) = (-1)^{n(G)} z(G)$  when each of the  $n(G)$  components of  $G$  is strongly connected.

The Theorem 3.4 also proves Theorem 1.1. It is obvious that the Berezin star product defined in (15) is local in the sense that  $\text{supp } C_j(f_1, f_2)$  is contained in  $\text{supp } f_1 \cap \text{supp } f_2$  for all  $j \geq 1$ . Since  $Q_\Gamma = 0$  if  $\Gamma$  is not strongly connected, we see that (15) defines a deformation quantization with separation of variables, namely it satisfies  $f \star h = f \cdot h$  and  $h \star g = h \cdot g$  for any locally defined holomorphic function  $f$ , antiholomorphic function  $g$  and an arbitrary function  $h$ . In particular, 1 is the unit in the star product.

**Proposition 3.8.** *The Berezin star product satisfies  $\overline{f_1 \star f_2} = \bar{f}_2 \star \bar{f}_1$ .*

*Proof.* We have

$$\begin{aligned} \overline{f_1 \star f_2}(x) &= \sum_{\Gamma = (V \cup \{f\}, E) \in \dot{\mathcal{G}}_{scn}} \frac{\det(A(\Gamma_-) - I)}{|\text{Aut}(\Gamma)|} h^{|E| - |V|} D_{\Gamma^T}(\bar{f}_2, \bar{f}_1) \Big|_x \\ &= \sum_{\Gamma = (V \cup \{f\}, E) \in \dot{\mathcal{G}}_{scn}} \frac{\det(A(\Gamma_-^T) - I)}{|\text{Aut}(\Gamma^T)|} h^{|E| - |V|} D_{\Gamma^T}(\bar{f}_2, \bar{f}_1) \Big|_x \\ &= \bar{f}_2 \star \bar{f}_1. \end{aligned}$$

Here  $\Gamma^T$  is the transpose of  $\Gamma$ , namely  $\Gamma^T$  is obtain by reversing all arrows in  $\Gamma$ . □

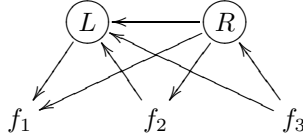


FIGURE 3. A graph  $\Gamma$  with 3 distinguished vertices

**Proposition 3.9.** *The Berezin star product is associative, namely  $f_1 \star (f_2 \star f_3) = (f_1 \star f_2) \star f_3$ .*

*Proof.* We have

$$(46) \quad f_1 \star f_2 = \sum_{\Gamma = (V \cup \{f\}, E) \in \dot{\mathcal{G}}^{ss}} Q_\Gamma h^{|E| - |V|} D_\Gamma(f_1, f_2).$$

As we are taking summation over pointed semistable graphs, this equation holds in a neighborhood of  $x$ .

Since the associativity is equivalent to

$$(47) \quad \sum_{j=0}^k C_j(f_1, C_{k-j}(f_2, f_3)) = \sum_{j=0}^k C_{k-j}(C_j(f_1, f_2), f_3),$$

it is enough to prove that for any two pointed semistable graphs  $L, R$  in  $\dot{\mathcal{G}}^{ss}$  as shown in Figure 3, the coefficients of any given stable graph  $\Gamma$  in  $Q_L D_L(f_1, Q_R D_R(f_2, f_3))$  and  $Q_R D_R(Q_L D_L(f_1, f_2), f_3)$  are equal. In fact, by Lemma 2.9 and Remark 2.10, the coefficients of a stable graph  $\Gamma$  in both terms are equal to

$$(48) \quad Q_L Q_R |\text{Aut}(L)| |\text{Aut}(R)| \frac{1}{|\text{Aut}(\Gamma)|}.$$

Note that  $\Gamma$  has three distinguished vertices labeled by  $f_1, f_2$  and  $f_3$ . An automorphism in  $\text{Aut}(\Gamma)$  should fix these three vertices. □

4. COMPUTATIONS OF  $R_k, Q_k, k \leq 3$ 

Fix a normal coordinate around  $x \in M$ . The covariant derivative of  $f$  satisfies

$$(49) \quad f_{;\alpha_1 \dots \alpha_p \beta} = \partial_\beta f_{;\alpha_1 \dots \alpha_p} - \sum_{i=1}^p \Gamma_{\beta \alpha_i}^\gamma f_{;\alpha_1 \dots \alpha_{i-1} \gamma \alpha_{i+1} \dots \alpha_p}.$$

We can use the above equation to write partial derivatives of  $f$  in terms of covariant derivatives of  $f$ .

**Example 4.1.** The formulae of  $R_k$  and  $Q_k$  in partial derivatives may be computed readily using (16), Theorem 3.2 and Theorem 3.4. We will convert them to the curvature-tensor expressions. Note that our convention of curvatures  $R_{i\bar{j}k\bar{l}}, R_{i\bar{j}}, \rho$  in [52] all differ by a minus sign with that of [19]. The following notations were introduced in [19].

$$(50) \quad L_{Ric} f := R_{i\bar{j}} f_{;\bar{j}\bar{i}}, \quad L_R f := R_{i\bar{j}k\bar{l}} f_{;\bar{j}\bar{l}\bar{k}}.$$

When  $k = 0$ ,  $R_0 = Q_0 = 1$ .

When  $k = 1$ , we have

$$\begin{aligned} R_1 f &= \left[ \bullet \curvearrowright_1 \right] + \frac{1}{2} [\textcircled{2}] = f_{i\bar{i}} + \frac{1}{2} g_{i\bar{i}j\bar{j}} f \\ &= f_{;\bar{i}\bar{i}} - \frac{1}{2} \rho f \\ &= \Delta f - \frac{1}{2} \rho f. \end{aligned}$$

$$\begin{aligned} Q_1 f &= \left[ \bullet \curvearrowright_1 \right] = f_{i\bar{i}} \\ &= \Delta f. \end{aligned}$$

When  $k = 2$ , we have

$$\begin{aligned} R_2 f &= \frac{1}{2} \left[ \bullet \curvearrowright_2 \right] + \frac{1}{2} \left[ \textcircled{2} \mid \bullet \curvearrowright_1 \right] + \frac{1}{2} \left[ \textcircled{2} \longleftarrow \bullet \right] + \frac{1}{2} \left[ \textcircled{2} \longrightarrow \bullet \right] \\ &\quad + \frac{1}{3} [\textcircled{3}] - \frac{3}{8} \left[ \circ \overset{2}{\curvearrowright} \circ \right] - \frac{1}{2} \left[ \textcircled{1} \overset{1}{\curvearrowright} \textcircled{1} \right] + \frac{1}{8} [\textcircled{2} \mid \textcircled{2}] \\ &= \frac{1}{2} f_{i\bar{i}j\bar{j}} + \frac{1}{2} g_{i\bar{i}j\bar{j}} f_{k\bar{k}} + \frac{1}{2} g_{i\bar{i}j\bar{j}\bar{k}} f_k + \frac{1}{2} g_{i\bar{i}j\bar{j}k} f_{\bar{k}} \\ &\quad + \left( \frac{1}{3} g_{i\bar{i}j\bar{j}k\bar{k}} - \frac{3}{8} g_{i\bar{j}k\bar{l}} g_{j\bar{l}\bar{k}} - \frac{1}{2} g_{i\bar{k}\bar{l}} g_{j\bar{j}l\bar{k}} + \frac{1}{8} g_{i\bar{i}j\bar{j}} g_{k\bar{k}l\bar{l}} \right) f \\ &= \frac{1}{2} \Delta^2 f - \frac{1}{2} L_{Ric} f - \frac{\rho}{2} \Delta f - \frac{1}{2} (\rho_{;\bar{k}} f_{;k} + \rho_{;k} f_{;\bar{k}}) \\ &\quad + \left( -\frac{1}{3} \Delta \rho - \frac{1}{24} |R|^2 + \frac{1}{6} |Ric|^2 + \frac{1}{8} \rho^2 \right) f. \end{aligned}$$

and

$$\begin{aligned} Q_2 f &= \frac{1}{2} \left[ \bullet \curvearrowright_2 \right] = \frac{1}{2} f_{i\bar{i}j\bar{j}} \\ &= \frac{1}{2} f_{;\bar{i}\bar{i}j\bar{j}} - \frac{1}{2} R_{i\bar{k}} f_{;\bar{k}\bar{i}} \\ &= \frac{1}{2} \Delta^2 f - \frac{1}{2} L_{Ric} f. \end{aligned}$$

When  $k = 3$ , we express  $Q_3 f$  in terms of the basis as used in Engliš [19].

$$(51) \quad \begin{aligned} \sigma_1 &= \Delta^3 f, \quad \sigma_2 = R_{i\bar{j}} (\Delta f)_{;\bar{j}\bar{i}}, \quad \sigma_3 = R_{i\bar{j}k\bar{l}} f_{;\bar{j}\bar{l}\bar{k}}, \\ \sigma_4 &= R_{i\bar{j};\bar{k}} f_{;\bar{j}\bar{i}k}, \quad \sigma_5 = R_{i\bar{j};k} f_{;\bar{j}\bar{l}\bar{k}}, \quad \sigma_6 = R_{i\bar{j}k\bar{l}} R_{j\bar{i}m\bar{k}} f_{;\bar{l}\bar{m}}, \\ \sigma_7 &= R_{i\bar{j}k\bar{l}} R_{j\bar{i}} f_{;\bar{l}\bar{k}}, \quad \sigma_8 = \rho_{;\bar{i}\bar{j}} f_{;\bar{j}\bar{i}}, \quad \sigma_9 = R_{i\bar{j}} R_{k\bar{l}} f_{;\bar{j}\bar{k}}. \end{aligned}$$

We will compute the coefficients  $c_i$ ,  $1 \leq i \leq 9$ , such that

$$(52) \quad Q_3 f = c_1 \sigma_1 + c_2 \sigma_2 + \cdots + c_9 \sigma_9.$$

There are 9 strongly connected pointed stable graphs of weight 3 in  $\dot{\mathcal{G}}_{scon}(3)$ .

$$(53) \quad \begin{aligned} \tau_1 &= \left[ \bullet \begin{array}{c} \curvearrowright \\ 3 \end{array} \right], & \tau_2 &= \left[ \textcircled{1} \begin{array}{c} \xrightarrow{1} \bullet \\ \xleftarrow{1} \end{array} \right], & \tau_3 &= \left[ \circ \begin{array}{c} \xrightarrow{2} \bullet \\ \xleftarrow{2} \end{array} \right], \\ \tau_4 &= \left[ \textcircled{1} \begin{array}{c} \xrightarrow{1} \bullet \\ \xleftarrow{2} \end{array} \right], & \tau_5 &= \left[ \textcircled{1} \begin{array}{c} \xrightarrow{2} \bullet \\ \xleftarrow{1} \end{array} \right], & \tau_6 &= \left[ \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \xrightarrow{1} \quad \xleftarrow{1} \\ \xrightarrow{2} \end{array} \right], \\ \tau_7 &= \left[ \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \textcircled{1} \\ \xrightarrow{1} \quad \xleftarrow{1} \\ \xrightarrow{1} \end{array} \right], & \tau_8 &= \left[ \textcircled{2} \begin{array}{c} \xrightarrow{1} \bullet \\ \xleftarrow{1} \end{array} \right], & \tau_9 &= \left[ \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \textcircled{1} \quad \textcircled{1} \\ \xrightarrow{1} \quad \xleftarrow{1} \\ \xrightarrow{1} \end{array} \right]. \end{aligned}$$

By Theorem 3.4, we have

$$(54) \quad Q_3 f = q_1 \tau_1 + q_2 \tau_2 + \cdots + q_9 \tau_9,$$

where

$$\begin{aligned} q_1 &= 1/6, & q_2 &= 0, & q_3 &= -1/4, & q_4 &= 0, & q_5 &= 0, \\ q_6 &= -1/2, & q_7 &= -1, & q_8 &= 1/2, & q_9 &= 0. \end{aligned}$$

We need to express each  $\tau_i$  as a linear combination of  $\sigma_i$ ,  $1 \leq i \leq 9$ . By a tedious but straightforward computation, we get

$$\begin{aligned} \tau_1 &= \sigma_1 + 3\sigma_2 + 2\sigma_3 + 2\sigma_4 + 2\sigma_5 + \sigma_6 + 4\sigma_7 + \sigma_8 - 2\sigma_9, \\ \tau_2 &= \sigma_2 + \sigma_7, & \tau_3 &= \sigma_3 + \sigma_6, & \tau_4 &= \sigma_4, & \tau_5 &= \sigma_5, \\ \tau_6 &= \sigma_6, & \tau_7 &= \sigma_7, & \tau_8 &= \sigma_6 + 2\sigma_7 + \sigma_8, & \tau_9 &= \sigma_9. \end{aligned}$$

Substituting into (54), we can get the coefficients in (52).

$$\begin{aligned} c_1 &= 1/6, & c_2 &= -1/2, & c_3 &= -1/12, & c_4 &= -1/3, & c_5 &= -1/3, \\ c_6 &= -1/12, & c_7 &= 2/3, & c_8 &= -2/3, & c_9 &= -1/3. \end{aligned}$$

All these values of  $R_k, Q_k$  computed here match the computations by Engliš [19].

From  $Q_k$ ,  $k \leq 3$ , we can get the invariant expressions for the first four coefficients of the Berezin star product (3).

$$\begin{aligned} C_0(f_1, f_2) &= f_1 f_2, \\ C_1(f_1, f_2) &= f_{1;\bar{i}} f_{2;i}, \\ C_2(f_1, f_2) &= \frac{1}{2} f_{1;\bar{i}\bar{j}} f_{2;ij}, \\ C_3(f_1, f_2) &= \frac{1}{6} f_{1;\bar{i}\bar{j}\bar{k}} f_{2;ijk} + \frac{1}{4} R_{i\bar{j}k\bar{l}} f_{1;\bar{i}\bar{k}} f_{2;jl} - \frac{1}{2} R_{i\bar{j}k\bar{l}} R_{j\bar{i}m\bar{k}} f_{1;\bar{m}} f_{2;l} \\ &\quad - R_{i\bar{j}k\bar{l}} R_{j\bar{i}\bar{k}} f_{1;\bar{k}} f_{2;l} - \frac{1}{2} \rho_{i\bar{j}} f_{1;\bar{i}} f_{2;j}. \end{aligned}$$

Note that around a normal coordinate system of  $x$ , we have  $f_{i_1, \dots, i_r}(x) = f_{;\bar{i}_1, \dots, \bar{i}_r}(x)$  and  $f_{\bar{i}_1, \dots, \bar{i}_r}(x) = f_{;i_1, \dots, i_r}(x)$  for any  $r \geq 1$ .

**Example 4.2.** We now describe how to compute  $R_3$  in terms of curvature tensors explicitly. The method works for any  $R_k$  or  $Q_k$  and for the Bergman kernel.

By Table 1, there are 46 pointed stable graphs of weight 3 in  $\dot{\mathcal{G}}(3)$ . Let  $\tau_i$ ,  $1 \leq i \leq 46$  be the corresponding Weyl invariants in terms of partial derivatives of metrics. They are also in one-to-one correspondence with a basis  $\sigma_i$ ,  $1 \leq i \leq 46$  of curvature tensors of weight 3. Each representative  $\sigma_i$  is determined up to an interchange of indices. By Ricci formula, the difference lies in the space of

strictly lower degree curvature tensors. (The definition of weight and order of a curvature tensor can be found in [38] or [52]).

It is relatively easy to express  $\sigma_i$  in terms of  $\tau_i$ . Namely we can obtain a  $46 \times 46$  square matrix  $M = [m_{ij}]_{1 \leq i, j \leq 46}$  of rational numbers, such that

$$(55) \quad \sigma_i = \sum_{j=1}^{46} m_{ij} \tau_j, \quad 1 \leq i \leq 46.$$

Let  $\tilde{M} = [\tilde{m}_{ij}]_{1 \leq i, j \leq 46}$  be the inverse matrix of  $M$ , then

$$(56) \quad \tau_i = \sum_{j=1}^{46} \tilde{m}_{ij} \sigma_j, \quad 1 \leq i \leq 46.$$

By Theorem 3.2 and (56), we finally get the curvature-tensor expression for  $R_3 f$ .

$$\begin{aligned} R_3 f &= \sum_{i=1}^{46} \frac{\det(A((\tau_i)_-) - I)}{|\text{Aut}(\tau_i)|} \tau_i \\ &= \sum_{i=1}^{46} \sum_{j=1}^{46} \tilde{m}_{ij} \frac{\det(A((\tau_i)_-) - I)}{|\text{Aut}(\tau_i)|} \sigma_i. \end{aligned}$$

We implemented the above procedure with the help of a computer and the final result of  $R_3 f$  matches with that computed by Engliš [19].

From Example 4.1 and the appendix, we have computed  $Q_k$ ,  $0 \leq k \leq 4$ , thus the first five terms of the Berezin star product (15).

**Example 4.3.** We now compute  $C_j^{BT}$ ,  $j \leq 3$ . By (10), we see that the coefficients of  $C_j^{BT}(f_1, f_2)$  equal to the coefficients in the asymptotic expansion of  $I^{-1}$ . The latter can be computed using (15).

$$(57) \quad I^{-1} = f - h \left[ \bullet \circlearrowleft_1 \right] + h^2 \left( \frac{1}{2} \left[ \bullet \circlearrowleft_2 \right] - \left[ \textcircled{1} \right] \right) + h^3 \left( -\frac{1}{6} \tau_1 + \tau_2 + \frac{1}{4} \tau_3 + \frac{1}{2} \tau_4 + \frac{1}{2} \tau_5 - \tau_9 \right) + O(h^4).$$

These  $\tau_i$  are graphs defined in (53).

Converting partial derivatives to covariant derivatives, we get the invariant expressions for the first four coefficients of the Berezin-Toeplitz star product (10).

$$\begin{aligned} C_0^{BT}(f_1, f_2) &= f_1 f_2, \\ C_1^{BT}(f_1, f_2) &= -f_{1;\bar{i}} f_{2;\bar{i}}, \\ C_2^{BT}(f_1, f_2) &= \frac{1}{2} f_{1;ij} f_{2;\bar{i}\bar{j}} + R_{i\bar{j}} f_{1;j} f_{2;\bar{i}}, \\ C_3^{BT}(f_1, f_2) &= -\frac{1}{6} f_{1;ijk} f_{2;\bar{i}\bar{j}\bar{k}} - R_{i\bar{j}} f_{1;jk} f_{2;\bar{i}\bar{k}} - \frac{1}{4} R_{i\bar{j}k\bar{l}} f_{1;jl} f_{2;\bar{i}\bar{k}} \\ &\quad - \frac{1}{2} R_{i\bar{j};\bar{k}} f_{1;jk} f_{2;\bar{i}} - \frac{1}{2} R_{i\bar{j};k} f_{1;j} f_{2;\bar{i}\bar{k}} - R_{i\bar{j}} R_{k\bar{i}} f_{1;j} f_{2;\bar{k}}. \end{aligned}$$

Compared to the Berezin star product, the holomorphic and antiholomorphic variables are swapped.

## 5. FEFFERMAN'S INVARIANTS

Fefferman [25] and Boutet de Monvel-Sjöstrand [8] proved the asymptotic expansion of the Bergman kernel of a strongly pseudoconvex domain in  $\mathbb{C}^n$  near the boundary. Nakazawa [44] obtained an

explicit formula for the first several coefficients in Fefferman's asymptotic expansion for bounded strictly pseudoconvex complete Reinhardt domains in  $\mathbb{C}^2$ .

In [19, 20], Engliš established a relation of Fefferman's invariants with the scalar invariants from the asymptotic expansion of the Laplace integral, and generalized Nakazawa's result to arbitrary Hartogs domains in  $\mathbb{C}^n$ .

In this section, we will apply Engliš' work to express Fefferman's invariants as graph invariants. First we state the following analogue of Theorem 3.1, again due to Engliš [19].

**Theorem 5.1.** (Engliš) *Let  $\Omega$  be a strongly pseudoconvex domain in  $\mathbb{C}^n$  with real analytic boundary. Then there is an asymptotic expansion for the Laplace integral*

$$\int_{\Omega} f(y) e^{-mD(x,y)} \frac{|\det g(x,y)|^2}{\det g(y)} dy \sim \frac{1}{m^n} \sum_{j \geq 0} m^{-j} R'_j(f)(x),$$

where  $\det g(x,y)$  is the almost analytic extension of  $\det g(x)$ , that is

$$(58) \quad \det g(x,y) = \det \left( \frac{1}{\pi} \frac{\partial^2 \Phi(x,y)}{\partial x_j \partial \bar{y}_k} \right),$$

and  $R'_j : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$  are explicit differential operators defined by

$$(59) \quad R'_j f(x) = \frac{1}{(\det g)^2} \sum_{k=j}^{3j} \frac{1}{k!(k-j)!} L^k(f(y) |\det g(x,y)|^2 S(x,y))|_{y=x},$$

where  $L$  is the (constant-coefficient) differential operator

$$L f(y) = g^{i\bar{j}}(x) \partial_i \partial_{\bar{j}} f(y)$$

and the function  $S(x,y)$  is given in Theorem 3.1.

We denote by  $K'_\alpha(x,y)$  the reproducing kernel of the weighted Bergman space of all holomorphic function on  $\Omega$  square-integrable with respect to the measure  $e^{-\alpha\Phi} dx$ . Locally,  $K'_\alpha(x,y)$  has an asymptotic expansion in a small neighborhood of the diagonal (see [19])

$$(60) \quad K'_\alpha(x,y) \sim e^{\alpha\Phi(x,y)} \det g(x,y) \sum_{k=0}^{\infty} B'_k(x,y) \alpha^{n-k}.$$

The corresponding Berezin transform is given by

$$(61) \quad I'_\alpha f(x) = \int_{\Omega} f(y) \frac{|K'_\alpha(x,y)|^2}{K'_\alpha(x,x)} e^{-\alpha\Phi(y)} dy,$$

which has an asymptotic expansion

$$(62) \quad I'_\alpha f(x) = \sum_{k=0}^{\infty} Q'_k f(x) \alpha^{-k}.$$

The following analogues of (40) and (32) still hold.

$$(63) \quad Q'_k f(x) = \sum_{j=0}^k \sum_{i=0}^{k-j} R'_j(B'_i(x,y) B'_{k-j-i}(y,x) f(y))|_{y=x} - \sum_{m=1}^k B'_m(x) Q'_{k-m} f(x),$$

$$(64) \quad B'_k(x) = - \sum_{\substack{i+j=k \\ i,j \geq 1}} B'_i(x) B'_j(x) - \sum_{\substack{\ell+i+j=k \\ 1 \leq \ell \leq k}} R'_\ell(B'_i(x,y) B'_j(y,x))|_{y=x}.$$

Let  $\tilde{K}(z,\zeta)$  be the (ordinary unweighted) Bergman kernel of the Hartogs domain

$$(65) \quad \tilde{\Omega} = \{z = (z_1, z_2) \in \Omega \times \mathbb{C}^d : \|z_2\|^2 < e^{-\Phi(z_1)}\}.$$

It was shown in [20] and [36] that

$$(66) \quad \tilde{K}((z_1, z_2), (\zeta_1, \zeta_2)) = \sum_{j=0}^{\infty} \frac{(j+d)!}{j! \pi^d} K'_{j+d}(z_1, \zeta_1) \langle z_2, \zeta_2 \rangle^k,$$

with the convergence uniform on compact subsets. Setting  $\zeta = z$ , we have

$$(67) \quad \tilde{K}(z_1, z_2) = \sum_{j=0}^{\infty} \frac{(j+d)!}{j! \pi^d} K'_{j+d}(z_1, z_1) \|z_2\|^{2j}.$$

By (60), the coefficients of this asymptotic expansion (Fefferman's invariants) are determined by  $B'_k$ . See [19] for a precise description of its behavior when  $z$  approaches the boundary of  $\tilde{\Omega}$ .

As pointed out by Engliš, in a normal coordinate around  $x$ ,  $R'_j$  in (59) simplifies to

$$(68) \quad R'_j f(x) = \sum_{k=j}^{2j} \frac{1}{k!(k-j)!} L^k(fS)|_{y=x},$$

which means that if we restrict to stable graphs, we do not need to consider their linear subgraphs when computing  $R'_\Gamma, Q'_\Gamma$  below

$$(69) \quad R'_k f = \sum_{\Gamma \in \dot{\mathcal{G}}(k)} R'_\Gamma \Gamma, \quad Q'_k f = \sum_{\Gamma \in \dot{\mathcal{G}}(k)} Q'_\Gamma \Gamma.$$

**Theorem 5.2.** *Let  $\Gamma = (V \cup \{f\}, E) \in \dot{\mathcal{G}}$ . Then we have*

$$(70) \quad R'_\Gamma = \frac{(-1)^{|V|}}{|\text{Aut}(\Gamma)|}$$

and

$$(71) \quad Q'_\Gamma = \begin{cases} \frac{(-1)^{|V|}}{|\text{Aut}(\Gamma)|} & \text{if } \Gamma \text{ is strongly connected,} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The formula for  $R'_\Gamma$  is obvious. The formula for  $Q'_\Gamma$  follows from (63), (70) and (74) by using the same argument in the proof of Theorem 3.4.  $\square$

**Corollary 5.3.** *Let  $r'_k = R'_k(1)$ . Then we have  $r'_0 = 1$  and when  $k \geq 1$  (see Remark 2.3)*

$$(72) \quad r'_k = \sum_{G \in \dot{\mathcal{G}}(k)} r'(G) G = \sum_{G \in \dot{\mathcal{G}}(k)} \frac{(-1)^{|V|}}{|\text{Aut}(G)|} G.$$

We may write  $B'_k$  as a summation over stable graphs.

$$(73) \quad B'_k(x) = \sum_{G \in \dot{\mathcal{G}}(k)} z'(G) \cdot G, \quad z'(G) \in \mathbb{Q}.$$

**Corollary 5.4.** *If  $G = (V, E) \in \dot{\mathcal{G}}$  is a disjoint union of connected subgraphs  $G = G_1 \cup \dots \cup G_n$ , then we have*

$$(74) \quad z'(G) = \begin{cases} \frac{(-1)^{|V|+n}}{|\text{Aut}(G)|}, & \text{if all } G_i \text{ are strongly connected,} \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* When  $G$  is strongly connected, we can use (64) and (70) to prove that

$$(75) \quad z'(G) = \frac{(-1)^{|V|+1}}{|\text{Aut}(G)|}.$$

In general, if  $G = G_1 \cup \dots \cup G_n$  is disjoint union of connected subgraphs and some  $G_i$  is not strongly connected, we can use the same argument of Proposition 3.3 to prove that  $z'(G) = 0$ . If all  $G_i$  are strongly connected, it follows from Lemma 2.9 and (70) that

$$(76) \quad z'(G) = \prod_{j=1}^n z'(G_j) / |\text{Sym}(G_1, \dots, G_n)|,$$

where  $\text{Sym}(G_1, \dots, G_m)$  denote the permutation group of these  $n$  connected subgraphs. So we conclude the proof of the formula (74).  $\square$



APPENDIX A. THE VALUE OF  $Q_4$ 

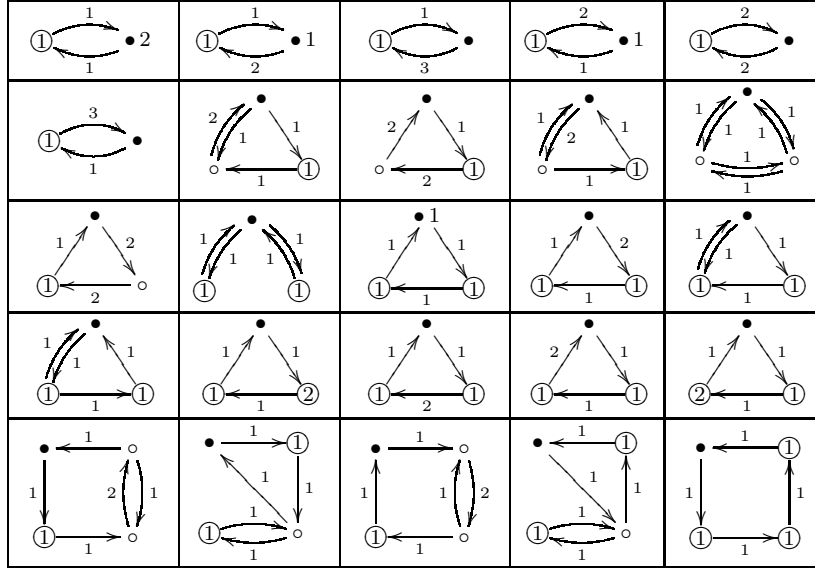
There are 36 strongly connected pointed stable graphs  $\Gamma$  in  $\dot{\mathcal{G}}_{scn}(4)$  such that  $Q_\Gamma \neq 0$ , which are listed in Table 2. There are 25 strongly connected pointed stable graphs  $\Gamma$  in  $\dot{\mathcal{G}}_{scn}(4)$  such that  $Q_\Gamma = 0$ , which are listed in Table 3. Thus we can use the method of Example 4.2 to get a curvature-tensor expression for  $Q_4$ .

TABLE 2.  $Q_\Gamma$  of  $\Gamma \in \dot{\Lambda}(4)$

1/24	-1/4	-1/12	-1/12	1/2	1/4
1/4	1/3	1/8	-1	-1	-1/2
-1	-1	-1/2	-1/4	-1/2	-1/2
-1	-1	-1/3	-1/2	-1/4	-1
-3/4	-3/4	-1	-1	-1	1
3/2	2	1	1	1	3/4

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TABLE 3.  $\Gamma \in \dot{\mathcal{G}}_{scon}(4)$  with  $Q_\Gamma = 0$ 

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